Math 255A' Lecture 12 Notes

Daniel Raban

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1 Reflexive Banach Spaces and Metrizability of the Unit Ball in the Weak* Topology

Today's lecture was given by a guest lecturer, Professor Dimitri Shlyakhtenko.

1.1 Reflexive spaces

Let X be a Banach space. For this lecture, we will denote the weak topology by $\sigma(X, X^*)$ and the weak*-topology by $\sigma(X^*, X)$. Lst time we proved the following theorem:

Theorem 1.1 (Alaoglu)**.** *$(X^*)_1$ is $\sigma(X^*, X)$ compact.*

Definition 1.1. X is **reflexive** if $X = X^{**}$.

Example 1.1. For $1 < p < \infty$, L^p and ℓ^p are reflexive.

Proposition 1.1. *$(X)_1 \subseteq (X^{**})_1$ is $\sigma(X^{**}, X^*)$-dense.*

Remark 1.1. $\sigma(X^{**}, X^*)|_X = \sigma(X, X^*)$.

Proof. Take B to be the closure of $(X)_1$ in the $\sigma(X^{**}, X)$ topology. Then $B \subseteq (X^{**})_1$ as $(X^{**})_1$ is closed. If $x^{**} \in (X^{**})_1 \setminus B$, then by Hahn-Banach (on $(X^{**}, \sigma * X^{**}, X^*)$), there exist an $x^* \in X^*$ and $\alpha \in \mathbb{R}$ such that $\operatorname{Re}\langle x, x^* \rangle < \alpha < \alpha + \varepsilon < \operatorname{Re}\langle x^*, x_0^{**} \rangle$ for all $x \in (X)_1$. So there is an $x^* \in X^*$ such that $\operatorname{Re}\langle x, x^* \rangle < 1 < 1 + \varepsilon < \operatorname{Re}\langle x^*, x_0^{**} \rangle$ for all $x \in X_0$. Then $|\langle x, x^* \rangle| = 1$ if $\|x\| \leq 1$, so $x^* \in (X^*)_1$. We now get

$$1 + \varepsilon < \operatorname{Re}\langle x^*, x_0^{**} \rangle \leq |\langle x^{,} x_0^{**} \rangle| \leq \|x_0^{**}\| \leq 1.$$

This is a contradiction. ∎

Theorem 1.2. *Let X be a Banach space. The following are equivalent:*

1. *X is reflexive.*

2. X^* *is reflexive.*

3. $\sigma(X^*, X) = \sigma(X^*, X^{**})$.

4. $(X)_1$ *is compact in* $\sigma(X, X^*)$.

Proof. (1) $\implies$ (3): This is because $X = X^{**}$.

(1) $\implies$ (4): If $X = X^{**}$, Then $(X)_1 = (X^{**})_1$. So $\sigma(X, X^*) = \sigma(X^{**}, X^*)$, the weak* topology on X^{**}. So $(X)_1$ is compact by Alaoglu's theorem.

(4) $\implies$ (1): Note that $\sigma(X^{**}, X^*)|_X = \sigma(X, X^*)$. Thus, if $(X)_1$ is $\sigma(X, X^*)$ compact, then $(X)_1$ is compact in $\sigma(X^{**}, X^*)$ as a subset of X^{**}. By the proposition, $(X)_1$ is $\sigma(X^*, X^{**})$-dense in X^{**}. And compact implies closed.

(3) $\implies$ (2): By Alaoglu's theorem, $(X^*)_1$ is $\sigma(X^*, X)$-compact. By assumption, $(X^*)_1$ is $\sigma(X^*, X^{**})$-compact. Now apply the argument of (4) $\implies$ (1) to X^*. So X^* is reflexive.

(2) $\implies$ (1): Observe that $(X)_1$ is norm-closed in X^{**} (because this is an isometric inclusion). Therefore, $(X)_1$ is $\sigma(X^{**}, X^{***})$-closed. Assuming (2), $(X)_1$ is $\sigma(X^{**}, X^*)$-closed. Bt the proposition, $(X)_1$ is $\sigma(X^{**}, X^*)$-dense in $(X^{**})_1$. So $(X)_1 = (X^{**})_1$. $\square$

1.2 Additional properties of reflexive spaces

Corollary 1.1. *Let* X *be a Banach space. If* $Y \subseteq X$ *is a closed subspace, then* Y *is reflexive.*

Proof. We have $(Y)_1 = Y \cap (X)_1$. So if X is reflexive, then $(Y)_1$ is (X, X^*)-compact. But then $\sigma(X, X^*)|_Y = \sigma(Y, Y^*)$ (check this using Hahn-Banach). So Y is reflexive. $\square$

Example 1.2. Can you embed $\ell^\infty \subseteq \ell^2$ isometrically? No. Since ℓ^∞ is not reflexive. What about embedding L^2 into L^∞?

Corollary 1.2. *Let* X *be a Banach space. If* X *is reflexive,* X *is* ***weakly sequentially complete****. That is, any* $\sigma(X, X^*)$*-Cauchy sequence has a* $\sigma(X, X^*)$*-limit.*

Proof. Suppose $\{x_n\}_n$ is weakly Cauchy: for all $x \in X^*$, $\{\langle x_n, x^* \rangle\}$ is Cauchy. Then it is bounded. The principle of uniform boundedness implies that there exists some M such that $|\langle x_n, x^* \rangle| \leq M$ for any $x^* \in (X^*)_1$. So $\|x_n\| \leq M$ for all n. Now by reflexivity, $M(X)_1$ is compact in $\sigma(X, X^*)$. So there exists a limit point $x \in M(X)_1$ such that $\langle x_n, x^* \rangle \to \langle x, x^* \rangle$ for all $x^* \in X^*$. Then $x_n \to x$ in the $\sigma(X, X^*)$-topology. $\square$

Example 1.3. Let $X = C([0, 1])$, and let

$$f_n = \begin{cases} -nx + 1 & x \in [0, 1/n] \\ 0 & x \in (1/n, 1]. \end{cases}$$

Then for any signed measure $\mu \in C([0, 1])^*$, $\langle f_n, \mu \rangle = \int f_n(t)\, d\mu(t) \to \mu(\{0\})$, but $f_n \not\to f$ weakly. So be careful.

Definition 1.2. A subspace $Y \subseteq X$ is called **proximal** if for all $x_0 \in X$, there exists some $y_0 \in Y$ such that $\|x_0 - y_0\| = \operatorname{dist}(Y, x_0)$

Corollary 1.3. *Let X be a Banach space. If X is reflexive, then any subspace $Y \subseteq X$ is proximal.*

Proof. The map $x \mapsto \|x - x_0\|$ is $\sigma(X, X^*)$-semicontinous. Then $Y \cap \{x : \|x - x_0\| \leq 2\operatorname{dist}(x_0, Y)\}$ is a $\sigma(X, X^*)$-compact set (by reflexivity). Semicontinuous functions achieve their minima. □

Proposition 1.2. *If $x^* \in X^*$, then $\ker x^*$ is proximal if and only if there is an $x \in (X)_1$ such that $\langle x, x^* \rangle = \|x^*\|$.*

Example 1.4. Let $L : [0,1] \to \mathbb{C}$ be $\int_0^{1/2} f\,dx - \int_{1/2}^1 f\,dx$. The norm of L is never achieved (you want a step function, but this is not continuous), so $\ker L$ is not proximal.

Theorem 1.3 (James, 60s)**.** *X is reflexive if and only if every closed hyperplane is proximal.*

1.3 Metrizability of the closed unit ball in the weak* topology

Theorem 1.4. *Let X be a Banach space. $(X^*)_1$ is $\sigma(X^*, X)$-metrizable if and only if X is separable.*

Proof. ($\Longleftarrow$): Assume X is separable. Let $\{x_n\}_n$ be a countable dense subset of X. Let $\mathbb{D} = \{z : |z| \leq 1\}$, and let $Y = \prod_{\mathbb{N}} \mathbb{D}$. Then the map $\tau_1^* \to X$ by $\tau(x^*) = \{\langle x^*, x_n \rangle\}_n$ gives a homeomorphism from $((X^*)_1, \sigma(X^*, X)) \to Y$.

($\Longrightarrow$): If $(X^*)_1$ is $\sigma(X^*, X)$-metrizable, then there are open sets $U_n \subseteq (X^*)_1$ with $0 \in U_n$ such that $\bigcap_n U_n = \{0\}$. By the definition of the weak* topology, there exist finites subsets F_n of X such that $\{x^* \in (X^*)_1 : |\langle x^*, x \rangle| < 1 \, \forall x \in F_n\} \subseteq U_n$. Let $F = \bigcup_n F_n$.

We claim that F is dense. Then $\overline{F} =^{\perp} (F^{\perp})$. So it is enough to prove that $F^{\perp} = \{0\}$. If $x^{\in} F^{\perp} \setminus \{0\}$, then for all $x \in F_n$,

$$0 = \left| \left\langle \frac{x^*}{\|x^*\|}, x \right\rangle \right| < 1 \implies \frac{x^*}{\|x^*\|} \in U_n \implies \frac{x^*}{\|x^*\|} = 0.$$

So $F^{\perp} = \{0\}$. □